Published by Institute of Physics Publishing for SISSA/ISAS

Received: *June 29, 2004*
Accepted: *July 24, 2004*

More on BPS solutions of $N = 2$, $D = 4$ gauged supergravity

Sergio L. Cacciatori

Dipartimento di Matematica dell'Università di Milano
Via Saldini 50, I-20133 Milano, Italy, and
INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
E-mail: cacciatori@mi.infn.it

Marco M. Caldarelli, Dietmar Klemm and Diego S. Mansi

Dipartimento di Fisica dell'Università di Milano
Via Celoria 16, I-20133 Milano, Italy, and
INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
E-mail: marco.caldarelli@mi.infn.it, dietmar.klemm@mi.infn.it, diego.mansi@mi.infn.it

Abstract: We deepen and refine the classification of supersymmetric solutions to $N = 2$, $D = 4$ gauged supergravity obtained in a previous paper. In the case where the Killing vector constructed from the Killing spinor is timelike, it is shown that the nonlinear partial differential equations determining the BPS solutions can be derived from a variational principle. The corresponding action enjoys a solution-generating $\mathrm{PSL}(2, \mathbb{R})$ symmetry. In certain subcases the system reduces to different known theories, like two-dimensional dilaton gravity or the dimensionally reduced gravitational Chern-Simons theory. We find new supersymmetric solutions including, among others, kinks that interpolate between two AdS_4 vacua, electrovac waves on anti-Nariai spacetimes, or generalized Robinson-Trautman solutions. In the case where the Killing vector is null, we obtain a complete classification. The one quarter and one half supersymmetric solutions are determined explicitely, and it is shown that the fraction of three quarters of supersymmetry cannot be preserved. Finally, the general lightlike configuration is uplifted to eleven-dimensional supergravity.

Keywords: Superstring Vacua, AdS-CFT and dS-CFT Correspondence, Black Holes, Supergravity Models.

Contents

1. Introduction

Supersymmetric solutions have played an important role in recent progress in string theory. This makes it desirable to obtain a systematic classification of BPS solutions to various supergravity theories. Apart from the seminal work by Tod [1], who wrote down all metrics admitting supercovariantly constant spinors in $N = 2$, $D = 4$ ungauged supergravity, progress in this direction has been made mainly during the last two years using the mathematical concept of G-structures [2]. The basic strategy is to assume the existence of at least one Killing spinor, and to construct differential forms as bilinears from this supercovariantly constant spinor. These forms obey several algebraic and differential equations that can be used to deduce the metric and the other bosonic supergravity fields. This formalism has been successfully applied to several supergravity theories in diverse dimensions [3, 4]. A common feature is that the Killing vector constructed from the Killing spinor is either timelike or lightlike, so that the solutions fall into two (partially overlapping) classes.

In this paper we will deepen and refine the classification of supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity obtained in [4].[1] The motivation for this is threefold: first of all, as pointed out above, having at hand a systematic approach to construct BPS solutions avoids the use of special ansätze that cover only a certain subclass of supersymmetric configurations. The second reason comes from the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence. In three dimensions there is a rich web of conformal field theories that have many interesting and physically relevant perturbative as well as non-perturbative properties. It might turn out that some of these CFTs have an AdS_4 dual. Then, supergravity vacua with less than maximal supersymmetry may have an interpretation on the CFT side as an expansion of the theory around non-zero vacuum expectation values of certain operators. Thirdly, Mathur et al. proposed recently a gravity picture of black hole microstates [6]. According to [6], these microstates are completely regular (i. e. , horizonless and nonsingular) supergravity solutions carrying the same charges and mass as the black hole. Coarse graining over the microstates (in a sense that is explained in [6]) yields the Bekenstein-Hawking entropy of the black hole. In the case of the D1-D5-P black hole, a set of such solutions has been constructed in [6, 7, 8]. It is plausible that, if the black hole under consideration is supersymmetric, then also the gravity microstates should preserve the same amount of supersymmetry. Thus, a classification of BPS solutions of $N = 2$, $D = 4$ gauged supergravity should be relevant for the construction of microstates for supersymmetric AdS black holes.

The remainder of this paper is organized as follows. In section 2, we discuss the case where the Killing vector $V^\mu = i\bar{\epsilon}\Gamma^\mu\epsilon$ obtained from the Killing spinor ϵ is timelike. The general form of the metric and the electromagnetic field strength was given in [4]. The geometry is characterized by some functions that satisfy highly nonlinear partial differential equations. Here, we shall reveal some of the mathematical structure behind these equations, and present many new solutions, which give rise to new supersymmetric supergravity configurations. In particular, we show that the differential equations follow from an action principle in three dimensions. This action enjoys a $\mathrm{PSL}(2,\mathbb{R})$ invariance, which can be used to generate new solutions from known ones. Quite surprisingly, it turns out that a certain

[1] For the inclusion of sources see [5].

subclass of solutions to the general equations is governed by the dimensionally reduced gravitational Chern-Simons action. The deeper significance of this intriguing fact is rather obscure and deserves further investigations. Among the new solutions that we shall present there are, among others, deformations of AdS_4, kink solutions that interpolate between two maximally supersymmetric AdS vacua, and charged supersymmetric generalizations of the Robinson-Trautman type geometries.
In section 3 we give a complete classification of the lightlike case, where the general supersymmetric solution is given by an electrovac AdS travelling wave [4], whose profile satisfies a generalized Siklos equation. It is shown that a configuration which admits a null Killing spinor, i. e. , a Killing spinor which can be used to construct a null Killing vector, is either one quarter or one half supersymmetric. The fraction three quarters of supersymmetry cannot be preserved in this case. For vanishing electromagnetic fields, the solution is always one quarter supersymmetric. The explicit form of the wave profile for the one half BPS case is given. We shall furthermore show that for half-supersymmetric solutions, the second Killing spinor gives rise to a timelike Killing vector, which implies that these waves are also solutions of the timelike case. Finally, the general lightlike geometry is lifted to a solution of eleven-dimensional supergravity.
We conclude in section 4 with some final remarks. The appendices contain our conventions and notations as well as some supplementary material.

2. The timelike case

Let us briefly recall the results of [4] for timelike V^μ (rewritten here in a slightly more compact form). The general BPS solution reads[2]

$$
\begin{aligned}
\mathrm{d}s^2 &= -\frac{4}{\ell^2 F\bar{F}}\left(\mathrm{d}t+\omega_i \mathrm{d}x^i\right)^2 + \frac{\ell^2 F\bar{F}}{4}\left[\mathrm{d}z^2 + e^{2\phi}\left(\mathrm{d}x^2+\mathrm{d}y^2\right)\right], \\
\mathcal{F} &= \frac{\ell^2}{4}F\bar{F}\left[V\wedge \mathrm{d}f + {}^*\left(V\wedge\left(\mathrm{d}g+\frac{1}{\ell}\mathrm{d}z\right)\right)\right],
\end{aligned} \tag{2.1}
$$

where $i=1,2$; $x^1=x$, $x^2=y$, and we defined $\ell F = 2i/(f-ig)$, with $f=\bar{\epsilon}\epsilon$ and $g=i\bar{\epsilon}\Gamma_5\epsilon$. Here, $1/\ell$ is the minimal coupling between the graviphoton and the gravitini, which is related to the cosmological constant by $\Lambda=-3\ell^{-2}$, and $\mathcal{F}$ denotes the electromagnetic field strength. The timelike Killing vector is given by $V=\partial_t$. The functions $\phi, F, \bar{F}$, that depend on x,y,z, are determined by the system

$$\Delta F + e^{2\phi}[F^3 + 3FF' + F''] = 0\,, \tag{2.2}$$

$$\Delta\phi + \frac{1}{2}e^{2\phi}[F' + \bar{F}' + F^2 + \bar{F}^2 - F\bar{F}] = 0\,, \tag{2.3}$$

$$\phi' - \mathrm{Re}(F) = 0\,, \tag{2.4}$$

where $\Delta = \partial_x^2 + \partial_y^2$, and a prime denotes differentiation with respect to z. (2.2) comes from the combined Maxwell equation and Bianchi identity, whereas (2.3) results from the

[2] We have chosen the conformal gauge for the two-metric h_{ij} appearing in [4].

integrability condition for the Killing spinor ϵ. Finally, the shift vector ω is obtained from[3]

$$\begin{aligned}\partial_z\omega_i &= \frac{\ell^4}{8}(F\bar{F})^2\epsilon_{ij}(f\partial_j g - g\partial_j f)\,,\\ \partial_i\omega_j - \partial_j\omega_i &= \frac{\ell^4}{8}(F\bar{F})^2 e^{2\phi}\epsilon_{ij}\left(f\partial_z g - g\partial_z f + \frac{2f}{\ell}\right),\end{aligned} \tag{2.5}$$

with $\epsilon_{12} = 1$.

2.1 Symmetries and properties of the equations

Before presenting new solutions of the timelike case, let us study some general properties of the system (2.2)–(2.4). First of all, we note that it is invariant under PSL$(2,\mathbb{R})$ transformations

$$z \to \frac{az+b}{cz+d}\,, \qquad ad - bc = 1\,, \tag{2.6}$$

if the fields transform according to

$$\begin{aligned}F &\to (cz+d)^2F - \partial_z(cz+d)^2\,,\\ \phi &\to \phi - 2\ln(cz+d)\,.\end{aligned} \tag{2.7}$$

This means that F has a connection-like transformation behaviour, whereas ϕ transforms like a Liouville field. In appendix B we show, using the supersymmetric Reissner-Nordström-Taub-NUT-AdS$_4$ geometry as an example, that this symmetry is nontrivial, i. e. , it can be used to generate new solutions.

If we introduce the complex coordinates $\zeta = x + iy$, $\bar{\zeta} = x - iy$, we see that the eqs. (2.2)–(2.4) enjoy an additional infinite-dimensional conformal symmetry

$$\zeta \to w(\zeta)\,, \qquad \bar{\zeta} \to \bar{w}(\bar{\zeta})\,, \qquad F \to F\,, \qquad \phi \to \phi - \frac{1}{2}\ln\left(\frac{dw}{d\zeta}\frac{d\bar{w}}{d\bar{\zeta}}\right), \tag{2.8}$$

where $w(\zeta)$ and $\bar{w}(\bar{\zeta})$ denote arbitrary holomorphic and antiholomorphic functions respectively. However, it is easy to see that, from the four-dimensional point of view, this represents merely a diffeomorphism that preserves the conformal gauge for the two-metric $e^{2\phi}(dx^2 + dy^2)$. Thus, unlike the PSL$(2,\mathbb{R})$ transformations above, this symmetry cannot be used to generate new solutions from known ones.

Decomposing F into its real and imaginary part, $F = A + iB$, we see that the real part of eq. (2.2) follows from (2.3) and (2.4), so that the remaining system is

$$\Delta B + e^{2\phi}\left[3\phi'^2 B - B^3 + 3\phi' B' + 3B\phi'' + B''\right] = 0\,, \tag{2.9}$$

$$\Delta\phi + \frac{1}{2}e^{2\phi}\left[2\phi'' + \phi'^2 - 3B^2\right] = 0\,, \tag{2.10}$$

together with $A = \phi'$. The equations (2.9), (2.10) can be derived from the action

$$S = \int \mathrm{d}^2x\,\mathrm{d}z\left[\nabla B\cdot\nabla\phi + \frac{1}{2}e^{2\phi}\left(B^3 + 2B'\phi' + 3B\phi'^2\right)\right]. \tag{2.11}$$

[3]It can be shown that the integrability conditions for (2.5) follow from the Maxwell equations.

This action is also invariant under the above $\mathrm{PSL}(2,\mathbb{R})$ transformations, with B transforming like the imaginary part of F, i. e. ,

$$B \to (cz+d)^2 B\,. \tag{2.12}$$

B transforms thus like a conformal field of weight two.

Multiplying eq. (2.2) by $\bar{F}$ and subtracting the complex conjugate yields the conservation law

$$\partial_i j_i + \rho' = 0\,, \tag{2.13}$$

with the current j_i and the "charge density" ρ given respectively by

$$\begin{aligned} j_i &= \frac{1}{2i}\left(\bar{F}\partial_i F - F\partial_i\bar{F}\right) = \phi'\partial_i B - B\partial_i\phi'\,, \\ \rho &= \frac{1}{2i}\left(\bar{\lambda}'\lambda'' - \lambda'\bar{\lambda}''\right) = e^{2\phi}\left[\left(\phi'^2 + B^2\right)B + \phi' B' - B\phi''\right], \end{aligned} \tag{2.14}$$

where $\lambda = \exp\int F\mathrm{d}z$. This current conservation is presumably related to the $\mathrm{PSL}(2,\mathbb{R})$ invariance of the action (2.11), although we did not check this explicitely.

In the "purely magnetic" case ($f = 0$), one has $B = 0$, so that the only equation to solve is

$$\Delta\phi + \frac{1}{2}e^{2\phi}\left[\phi'^2 + 2\phi''\right] = 0\,. \tag{2.15}$$

This is similar to the continuous ($\mathrm{SU}(\infty)$) Toda equation [9] (or heavenly equation)

$$\Delta\phi + \frac{1}{2}\partial_z^2 e^{2\phi} = 0\,, \tag{2.16}$$

which determines self-dual Einstein metrics that admit at least one rotational Killing vector [9].

In the "purely electric" case ($g = 0$) we have $A = 0$, and thus ϕ is independent of z. (2.10) implies then that also B does not depend on z, and the equations (2.9), (2.10) reduce to

$$\Delta B - e^{2\phi}B^3 = 0\,, \tag{2.17}$$

$$\Delta\phi - \frac{3}{2}e^{2\phi}B^2 = 0\,. \tag{2.18}$$

These equations follow from the two-dimensional dilaton gravity action

$$S = \int \mathrm{d}^2x\sqrt{g}\left[BR + B^3\right], \tag{2.19}$$

if we use the conformal gauge $g_{ij}\mathrm{d}x^i\mathrm{d}x^j = e^{2\phi}(\mathrm{d}x^2 + \mathrm{d}y^2)$. However, the equations of motion following from (2.19) contain also the constraints $\delta S/\delta g^{ij} = 0$ (whose trace yields (2.17)), and therefore they are more restrictive than the system (2.17), (2.18). Of course, every solution that extremizes the action (2.19) is a solution of our system, but not vice versa. It is interesting to note that (2.19) is similar to the action that arises from Kaluza-Klein reduction of the three-dimensional gravitational Chern-Simons term [10], with the only difference that here B is a fundamental field, whereas the field arising in [10] is the curl of a vector potential.

JHEP07(2004)061

We can actually obtain exactly the model considered in [10] by allowing for z-dependent A instead of setting $A = 0$, i. e. , we set $F = A(z) + iB(x,y,z)$. From (2.4) one obtains then

$$\phi = \int A(z)\,\mathrm{d}z + \gamma(x,y)\,, \tag{2.20}$$

with $\gamma(x,y)$ denoting a function of x,y alone. If we define

$$\beta(x,y,z) \equiv B \exp \int A\,dz\,, \tag{2.21}$$

eq. (2.3) implies that β^2 separates into a part that depends only on z and a function of x,y,

$$\beta^2(x,y,z) = Q(z) + R(x,y)\,.$$

Let us consider the case $Q = 0$. (2.3) yields then

$$\begin{aligned} \Delta\gamma + \frac{1}{2}e^{2\gamma}\left(k - 3\beta^2\right) &= 0\,, \\ e^{2\int A\,dz}\left(2A' + A^2\right) &= k\,, \end{aligned} \tag{2.22}$$

where k denotes an arbitrary constant. The latter equation is solved by

$$A = \frac{2az + b}{az^2 + bz + c}\,, \tag{2.23}$$

with a,b,c being real integration constants obeying $4ac - b^2 = k$. Finally, from (2.9) one obtains

$$\Delta\beta + e^{2\gamma}\left(k\beta - \beta^3\right) = 0\,. \tag{2.24}$$

We see that the equations (2.22), (2.24) represent generalizations of (2.18) and (2.17) respectively, which are recovered in the case $k = 0$.[4] What comes as a surprise is that (2.22) and (2.24) follow from the dimensionally reduced gravitational Chern-Simons action

$$S = \int \mathrm{d}^2x\sqrt{g}\left[\beta R + \beta^3\right]\,, \tag{2.25}$$

if we use the conformal gauge $g_{ij}\mathrm{d}x^i\mathrm{d}x^j = e^{2\gamma}(\mathrm{d}x^2 + \mathrm{d}y^2)$. Note that in (2.25) β is not a fundamental field, rather it is the curl of a vector potential, $\sqrt{g}\epsilon_{ij}\beta = \partial_i A_j - \partial_j A_i$. Did we vary β instead of A_i in the action, we would obtain the equations (2.17), (2.18), i. e. , the case $k = 0$. Like before, the equations of motion following from the action (2.25) are slightly stronger than our system, which does not include the traceless part of the constraints $\delta S/\delta g^{ij} = 0$.

We will now present some solutions of the system (2.2)–(2.4), which give rise to new BPS states of $N = 2$, $D = 4$ gauged supergravity.

[4]Actually also for $k = 0$ A can still depend on z, but in this case $A(z)$ is related to $A = 0$ by a $\mathrm{PSL}(2,\mathbb{R})$ transformation (2.6), (2.7). To see this, one observes that for $k = 0$ one has $2A' + A^2 = 0$ and thus $A = 2/(z + \tilde{b})$, where $\tilde{b}$ denotes an arbitrary integration constant. Now one performs a $\mathrm{PSL}(2,\mathbb{R})$ transformation with $d = c\tilde{b}$. The transformed A will then vanish. Note that it is no more possible to transform A to zero if $k \neq 0$.

JHEP07(2004)061

2.2 "Purely electric" solutions ($g = 0$)

First, we consider the case $g = 0$ and assume that B and ϕ depend only on x. In the coordinates system (2.1), the $g = 0$ condition corresponds to have a vanishing magnetic field, and therefore we shall refer to these solutions as the "purely electric" ones. If we use the ansatz $B = x^\alpha$, equations (2.17) and (2.18) are satisfied for $\alpha = 2$ or $\alpha = -1/3$. The former value of α yields the Petrov type-I solution obtained in [4], whereas for the latter one we get (after solving eq. (2.5) for ω) the four-dimensional geometry

$$\mathrm{d}s^2 = -\frac{4x^{2/3}}{\ell^2}\left(\mathrm{d}t - \frac{\ell^2}{6x^{4/3}}\mathrm{d}y\right)^2 + \frac{\ell^2}{4x^{2/3}}\mathrm{d}z^2 + \frac{\ell^2}{9x^2}(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \tag{2.26}$$

and the electromagnetic field strength

$$\mathcal{F} = -\frac{2}{3\ell x^{2/3}}\mathrm{d}t \wedge \mathrm{d}x\,. \tag{2.27}$$

Introducing the new coordinates

$$r = 2x^{1/3}\,, \qquad u = t\,, \qquad v = \frac{8}{3\ell^2}y\,,$$

the solution can be written in the form

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\ell^2}{r^2}\left[-\frac{r^4}{\ell^4}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}r^2 + \mathrm{d}z^2\right], \\ \mathcal{F} &= -\frac{1}{\ell}\mathrm{d}u \wedge \mathrm{d}r\,. \end{aligned} \tag{2.28}$$

This represents an electrovac AdS travelling wave; in section 3 we will show that it is also a solution of the lightlike case and that it has two Killing spinors, one that gives the timelike Killing vector $V = \partial_u$ and another that yields the lightlike Killing vector $U = \partial_v$. In addition to U and V, the geometry (2.28) admits the Killing vectors

$$\begin{aligned} Z &= \partial_z\,, \qquad K = u\partial_z - z\partial_v\,, \\ D &= u\partial_u - z\partial_z - r\partial_r - 3v\partial_v\,, \end{aligned} \tag{2.29}$$

which obey the commutation relations

$$\begin{aligned} [D, V] &= -V\,, \qquad [D, Z] = Z\,, \qquad [D, U] = 3U\,, \\ [K, V] &= -Z\,, \qquad [K, Z] = U\,, \qquad [D, K] = 2K\,. \end{aligned}$$

The isometry group acts transitively on the spacetime, which is thus homogeneous. A computation of the Weyl scalars shows that its Petrov type is N.

2.3 "Purely magnetic" solutions ($f = 0$)

When $f = 0$, the function $F = A$ is real, and the equations reduce to the system (2.22), (2.23) and (2.24), with $\beta = 0$ (which is a simple solution of eq. (2.24)). These are "purely magnetic" solutions, in the sense that in these coordinates the electric field vanishes, and are easily seen to be also static. eq. (2.22) becomes then the Liouville equation

$$\Delta\gamma + \frac{k}{2}e^{2\gamma} = 0\,, \tag{2.30}$$

which describes the metrics on euclidean 2-manifolds with constant curvature k.

Let us consider first the case $a = 0$ in (2.23). Without loss of generality we set $b = 1$, $c = 0$, so that $k = -1$ and $A = 1/z$. As a solution of the Liouville equation we choose $e^{2\gamma} = 2/x^2$. This yields the Bertotti-Robinson type $\mathrm{AdS}_2 \times \mathbb{H}^2$ spacetime, with a purely magnetic Maxwell field,

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{4z^2}{\ell^2}\mathrm{d}t^2 + \frac{\ell^2}{4z^2}\mathrm{d}z^2 + \frac{\ell^2}{2x^2}(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \\ \mathcal{F} &= -\frac{\ell}{2x^2}\mathrm{d}x \wedge \mathrm{d}y\,. \end{aligned} \tag{2.31}$$

In [11], this configuration was shown to preserve half of the supersymmetry, and to admit an $osp(2|2) \oplus so(2,1) \cong su(1,1|1) \oplus su(1,1)$ isometry superalgebra.

For a different from zero we can set without loss of generality $a = 1/\ell$, $b = 0$, $c = k\ell/4$, so that

$$A = \frac{2z}{z^2 + \frac{k\ell^2}{4}}\,. \tag{2.32}$$

If $k = 0$ we have $A = 2/z$ and $\gamma(x, y)$ is harmonic, so that the 2-manifold with metric $e^{2\gamma}(dx^2 + dy^2)$ is flat. The choice $\gamma = 0$ leads to the maximally supersymmetric AdS_4 vacuum solution,

$$ds^2 = -\frac{z^2}{\ell^2}\mathrm{d}t^2 + \frac{\ell^2}{z^2}\mathrm{d}z^2 + z^2(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \tag{2.33}$$

$$\mathcal{F} = 0\,. \tag{2.34}$$

As was shown in [4], this is also the only configuration with maximal supersymmetry.

For $k < 0$ the solutions of the Liouville equation (2.30) are classified according to their monodromy (cf. e. g. [12]). If we introduce polar coordinates (r, σ) on the (x, y)-plane, we have the following σ-independent solutions [12]:

- **Elliptic monodromy:**

$$e^{2\gamma} = -\frac{2m^2}{kr^2\sinh^2(m\ln r)}\,, \tag{2.35}$$

- **Parabolic monodromy:**

$$e^{2\gamma} = -\frac{2}{kr^2\ln^2 r}\,, \tag{2.36}$$

- **Hyperbolic monodromy:**

$$e^{2\gamma} = -\frac{2m^2}{kr^2\sin^2(m\ln r)}\,. \tag{2.37}$$

Here, m is a constant related to the Liouville-momentum. The corresponding supergravity solution is then given by

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{z}{\ell} + \frac{k\ell}{4z}\right)^2 \mathrm{d}t^2 + \frac{\mathrm{d}z^2}{\left(\frac{z}{\ell} + \frac{k\ell}{4z}\right)^2} + z^2 e^{2\gamma}(\mathrm{d}x^2 + \mathrm{d}y^2)\,, \\ \mathcal{F} &= \frac{k\ell}{4} e^{2\gamma} \mathrm{d}x \wedge \mathrm{d}y\,. \end{aligned} \tag{2.38}$$

In the case of elliptic monodromy and $m = 1$, this reduces to the configuration found in [13],[5] which preserves one quarter of the supersymmetry and admits an $s(2) \oplus su(1,1)$ isometry superalgebra [11].[6] (2.38) represents an extremal black hole with event horizon at $z = \sqrt{-k}\ell/2$.

For $k > 0$ the four-dimensional metric and gauge field are again given by (2.38), with the Liouville field

$$e^{2\gamma} = \frac{2m^2}{kr^2 \cosh^2(m \ln r)}\,. \tag{2.39}$$

Introducing the new coordinates θ, φ by $r^m = \tan\theta/2$, $m\sigma = \varphi$, we get

$$e^{2\gamma}(\mathrm{d}r^2 + r^2\mathrm{d}\sigma^2) = \frac{2}{k}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\,. \tag{2.40}$$

As σ is identified modulo 2π, we see that $\varphi \sim \varphi + 2\pi m$, so that the effect of the parameter m is to introduce a conical singularity ($0 < m < 1$) or an excess angle ($m > 1$) on the north- and south pole of the two-sphere. For $m = 1$ there is no singularity, and the solution reduces to the one-quarter supersymmetric magnetic monopole found by Romans [17]. The magnetic charge of the solution reads

$$Q_m = \frac{1}{4\pi}\int \mathcal{F} = \frac{m\ell}{2}\,. \tag{2.41}$$

If $m \neq 1$, there is a magnetic fluxline that passes through $x = y = 0$, which causes the magnetic charge to be different from the value $Q_m = \ell/2$ of Romans' solution.

2.4 Solutions with $F = A(z) + iB(x, y, z)$.

If we allow only for a z-dependence for A, then the conformal factor ϕ of the transverse two-metric is given by (2.3), where the function $\gamma(x, y)$ is an integration constant. In two cases the equations can be further simplified, depending on the form of the function β defined in (2.21). The first case is when β depends only on (x, y), and the second when $\beta = \beta(z)$.

2.4.1 The case $\beta = \beta(x, y)$

The simplest case of vanishing β has been analyzed in section 2.3. By relaxing this condition and allowing for an (x, y)-dependence in β, the BPS configurations are described by the system (2.22), (2.23) and (2.24). Interesting solutions of these equations can be found if

[5] For generalizations to five dimensions see [14, 15].

[6] $s(2)$ denotes the superalgebra introduced by E. Witten to formulate supersymmetric quantum mechanics [16].

we choose $a = 1$, $b = c = 0$ in (2.23), so that $k = 0$ and $A = 2/z$. Eqs. (2.22) and (2.24) reduce to

$$\Delta\gamma - \frac{3}{2}e^{2\gamma}\beta^2 = 0\,, \tag{2.42}$$

$$\Delta\beta - e^{2\gamma}\beta^3 = 0\,. \tag{2.43}$$

The form of A reminds us of AdS_4, which can be recovered setting $\beta = 0$. For this reason the spacetimes below can be considered as modifications of AdS_4. The important point is that these solutions are governed by the same set of equations which describes "purely electric" solutions of section 2.2, as can be checked by comparing the equations above with (2.18) and (2.17). In section 2.2 we found two solutions: $B = x^2$ and $B = x^{-1/3}$ which yield spacetimes of Petrov type-I and N respectively. One can now verify that there exist two analogous solutions setting $\beta = kx^2$ or $\beta = kx^{-1/3}$, where k is an arbitrary real constant, which we set equal to 2 for convenience. In the first case the related BPS spacetime and Maxwell field are

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{z^4}{\ell^2(z^2+x^4)}\mathrm{d}t^2 + \frac{\ell^2(z^2+x^4)}{z^4}\mathrm{d}z^2 + \frac{\ell^2(z^2+x^4)}{2x^6}\mathrm{d}x^2 + \\
&\quad + \frac{2z^2(z^2-6x^4)}{3x^3(z^2+x^4)}\mathrm{d}t\,\mathrm{d}y + \frac{7\ell^2(z^4+6z^2x^4-9x^8)}{18x^6(z^2+x^4)}\mathrm{d}y^2\,, \\
\mathcal{F} &= \frac{1}{(z^2+x^4)^2}\Big[-\frac{2}{\ell}xz^2(z^2-x^4)\mathrm{d}t\wedge\mathrm{d}x - \frac{2}{\ell}x^6 z\,\mathrm{d}t\wedge\mathrm{d}z + \\
&\quad + \frac{14}{3}\ell x^3 z\,\mathrm{d}y\wedge\mathrm{d}z - \frac{7\ell}{6x^2}(z^2-x^4)(z^2-3x^4)\mathrm{d}x\wedge\mathrm{d}y\Big]\,,
\end{aligned} \tag{2.44}$$

and in the second case we have

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{z^4x^{2/3}}{\ell^2(1+z^2x^{2/3})}\mathrm{d}t^2 + \frac{2z^2}{3x^{2/3}}\mathrm{d}t\,\mathrm{d}y + \frac{\ell^2(1+z^2x^{2/3})}{z^4x^{2/3}}\mathrm{d}z^2 + \frac{\ell^2(1+z^2x^{2/3})}{9x^2}\mathrm{d}x^2\,, \\
\mathcal{F} &= -\frac{2zx^{1/3}}{\ell(1+z^2x^{2/3})^2}\mathrm{d}t\wedge\mathrm{d}z - \frac{z^2(1-z^2x^{2/3})}{3\ell x^{2/3}(1+z^2x^{2/3})^2}\mathrm{d}t\wedge\mathrm{d}x\,.
\end{aligned} \tag{2.45}$$

A calculation of the Weyl scalars shows that the two spacetimes, as before, are of Petrov type-I and Petrov type N respectively. We finally stress the fact that these solutions can also be obtained from the "purely electric" solutions of section 2.2 by an appropriate $PSL(2,\mathbb{R})$ transformation.

2.4.2 Kink solutions and generalizations

More general solutions can be obtained in the $\beta = \beta(x, y)$ case. As we mentioned above, the system (2.22)-(2.24) follows from the dimensionally reduced gravitational Chern-Simons action, which (for $k > 0$) admits the "kink" solution [10]

$$\gamma = -2\ln\cosh\frac{\sqrt{k}}{2}X\,, \qquad \beta = \sqrt{k}\tanh\frac{\sqrt{k}}{2}X\,, \tag{2.46}$$

where the coordinate X is related to x by

$$x = \frac{1}{\sqrt{k}}\sinh\frac{\sqrt{k}}{2}X\cosh\frac{\sqrt{k}}{2}X + \frac{X}{2}\,. \tag{2.47}$$

Let us assume $a \neq 0$ in (2.23) and shift $z \to z - b/2a$. This yields

$$F = \frac{2(z + in \tanh \frac{\sqrt{k}}{2} X)}{z^2 + n^2}\,, \tag{2.48}$$

where we defined $n = \sqrt{k}/2a$. One can now solve (2.5) to determine ω_i. Finally, by rescaling $X \to X/\sqrt{k}$, $y \to y/\sqrt{k}$, $z \to 2nz/\ell$, $t \to \ell t/2n$, we can effectively set $k = 1$, $n = \ell/2$ in the supergravity solution, which reads

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{1}{\ell^2}\frac{(z^2 + \frac{\ell^2}{4})^2}{z^2 + \frac{\ell^2}{4}\tanh^2\frac{X}{2}}\left[\mathrm{d}t + \left(\frac{\ell^3}{4(z^2 + \frac{\ell^2}{4})\cosh^4\frac{X}{2}} - \frac{\ell}{\cosh^2\frac{X}{2}}\right)\mathrm{d}y\right]^2 + \\
&\quad + \ell^2\frac{z^2 + \frac{\ell^2}{4}\tanh^2\frac{X}{2}}{(z^2 + \frac{\ell^2}{4})^2}\mathrm{d}z^2 + \left(z^2 + \frac{\ell^2}{4}\tanh^2\frac{X}{2}\right)\left(\mathrm{d}X^2 + \frac{\mathrm{d}y^2}{\cosh^4\frac{X}{2}}\right), \\
\mathcal{A} &= \frac{1}{2}\frac{z^2 + \frac{\ell^2}{4}}{z^2 + \frac{\ell^2}{4}\tanh^2\frac{X}{2}}\tanh\frac{X}{2}\,\mathrm{d}t\,.
\end{aligned} \tag{2.49}$$

Asymptotically for $X \to \pm\infty$ the gauge field goes to zero and the metric approaches

$$\mathrm{d}s^2 \to -\left(\frac{z^2}{\ell^2} + \frac{1}{4}\right)\left[\mathrm{d}t \mp \frac{\ell}{u}\mathrm{d}y\right]^2 + \frac{\mathrm{d}z^2}{\frac{z^2}{\ell^2} + \frac{1}{4}} + \ell^2\left(\frac{z^2}{\ell^2} + \frac{1}{4}\right)\frac{\mathrm{d}u^2 + \mathrm{d}y^2}{u^2}\,, \tag{2.50}$$

where we defined the new coordinate $u = \pm e^{\pm X}/4$. Eq. (2.50) is simply AdS_4 written in nonstandard coordinates [18], so that the "kink" solution (2.49) interpolates between two AdS vacua at $X = \pm\infty$.

Grumiller and Kummer were able to write down the most general solution of (2.25), using the fact that the dimensionally reduced gravitational Chern-Simons theory can be written as a Poisson-sigma model with four-dimensional target space and degenerate Poisson tensor of rank two [19]. This solution is given by [19]

$$\begin{aligned}
\gamma &= -2\ln\cosh\frac{\sqrt{k}}{2}X + \frac{1}{2}\ln(1+\delta)\,, \\
\delta &= \left(\frac{8\mathcal{C}}{k^2} - 1\right)\cosh^4\frac{\sqrt{k}}{2}X\,, \\
\beta &= \sqrt{k}\tanh\frac{\sqrt{k}}{2}X\,,
\end{aligned} \tag{2.51}$$

where the coordinate X is related to x by

$$\frac{dx}{dX} = \frac{\cosh^2\frac{\sqrt{k}}{2}X}{1+\delta}\,,$$

and $\mathcal{C}$ denotes an integration constant.[7] In the special case $8\mathcal{C} = k^2$ we recover the kink solution considered above. As before, one can now determine F and ω_i corresponding to (2.51). This gives rise to new BPS supergravity solutions generalizing (2.49).

[7] More precisely, $\mathcal{C}$ and k are the Casimir functions of the Poisson sigma model that can be interpreted respectively as energy and charge [19].

Using the same coordinates as in (2.49) we find

$$
\begin{aligned}
\mathrm{d}s^2 &= -\frac{1}{\ell^2}\frac{(z^2+\frac{\ell^2}{4})^2}{z^2+\frac{\ell^2}{4}\tanh^2\frac{X}{2}}\left[\mathrm{d}t+\left(\frac{\ell^3(1+\delta)}{4(z^2+\frac{\ell^2}{4})\cosh^4\frac{X}{2}}-\frac{\ell}{\cosh^2\frac{X}{2}}\right)\mathrm{d}y\right]^2+\\
&\quad+\ell^2\frac{z^2+\frac{\ell^2}{4}\tanh^2\frac{X}{2}}{(z^2+\frac{\ell^2}{4})^2}\mathrm{d}z^2+\frac{(z^2+\frac{\ell^2}{4}\tanh^2\frac{X}{2})}{1+\delta}\left(\mathrm{d}X^2+\frac{(1+\delta)^2}{\cosh^4\frac{X}{2}}\mathrm{d}y^2\right),\\
\mathcal{A} &= \frac{1}{2}\frac{z^2+\frac{\ell^2}{4}}{z^2+\frac{\ell^2}{4}\tanh^2\frac{X}{2}}\tanh\frac{X}{2}\,\mathrm{d}t+\delta_0\frac{\ell^3}{8}\frac{\tanh\frac{X}{2}}{z^2+\frac{\ell^2}{4}\tanh^2\frac{X}{2}}\,\mathrm{d}y\,. \qquad (2.52)
\end{aligned}
$$

with $\delta_0=\frac{8\mathcal{C}}{k^2}-1$.

If $\delta_0\le 0$ then the metric is well-defined only in the region $-\bar{X}\le X\le\bar{X}$ where $\cosh(\bar{X}/2)=-1/\delta_0$ and $\delta_0\ge -1$.

If $\delta_0\ge 0$ we can take $X\to\pm\infty$ so that the metric becomes

$$
\mathrm{d}s^2=-\left(\frac{z^2}{\ell^2}+\frac{1}{4}\right)\left[\mathrm{d}t\mp\frac{\ell}{u}\mathrm{d}y+\frac{l^3\delta_0}{4z^2+\ell^2}\mathrm{d}y\right]^2+\frac{\mathrm{d}z^2}{\frac{z^2}{\ell^2}+\frac{1}{4}}+\ell^2\left(\frac{z^2}{\ell^2}+\frac{1}{4}\right)\left[\frac{\mathrm{d}u^2}{\delta_0u^4}+\delta_0\mathrm{d}y^2\right],
$$

where again $u=\pm e^{\pm X}/4$, and the gauge field asymptotes to

$$
\mathcal{A}=\frac{\delta_0\ell}{8}\frac{\mathrm{d}y}{\frac{z^2}{\ell^2}+\frac{1}{4}}\,. \qquad (2.53)
$$

Note that all these solutions are defined only for $k\ge 0$. One can now verify that the domain of the parameter k can be extended also to the negative region. Setting $\sqrt{k}=i\eta$, in general one has the following functions

$$
\begin{aligned}
\gamma &= -2\ln\cos\frac{\eta}{2}X+\frac{1}{2}\ln(1+\delta)\,,\\
\delta &= \left(\frac{8\mathcal{C}}{\eta^4}-1\right)\cos^4\frac{\eta}{2}X\,,\\
\beta &= -\eta\tan\frac{\eta}{2}X\,. \qquad (2.54)
\end{aligned}
$$

This yields

$$
F=\frac{2(z-in\tan\frac{\eta}{2}X)}{z^2-n^2}\,, \qquad (2.55)
$$

where we defined $n\equiv\eta/2a$. As before we can set, after a diffeomorphism, $k=-1$ and $n=\ell/2$, so that the solution reads

$$
\begin{aligned}
\mathrm{d}s^2 &= -\frac{1}{\ell^2}\frac{(z^2-\frac{\ell^2}{4})^2}{z^2+\frac{\ell^2}{4}\tan^2\frac{X}{2}}\left[\mathrm{d}t-\left(\frac{\ell^3(1+\delta)}{4(z^2-\frac{\ell^2}{4})\cos^4\frac{X}{2}}+\frac{\ell}{\cos^2\frac{X}{2}}\right)\mathrm{d}y\right]^2+\\
&\quad+\ell^2\frac{z^2+\frac{\ell^2}{4}\tan^2\frac{X}{2}}{(z^2-\frac{\ell^2}{4})^2}\mathrm{d}z^2+\frac{(z^2+\frac{\ell^2}{4}\tan^2\frac{X}{2})}{1+\delta}\left(\mathrm{d}X^2+\frac{(1+\delta)^2}{\cos^4\frac{X}{2}}\mathrm{d}y^2\right),\\
\mathcal{A} &= -\frac{1}{2}\frac{z^2-\frac{\ell^2}{4}}{z^2+\frac{\ell^2}{4}\tan^2\frac{X}{2}}\tan\frac{X}{2}\,\mathrm{d}t+\delta_0\frac{\ell^3}{8}\frac{\tan\frac{X}{2}}{z^2+\frac{\ell^2}{4}\tan^2\frac{X}{2}}\,\mathrm{d}y\,. \qquad (2.56)
\end{aligned}
$$

2.4.3 The case $\beta = \beta(z)$

In this case $B(x, y, z)$ must be a function of the coordinate z alone, and the system of equations which describes this set of solutions is

$$\begin{aligned} B'' + 3\,(AB)' + B\left(3A^2 - B^2\right) &= 0\,, \\ \check{\mathcal{R}}(\gamma) &= k\,, \\ e^{2\int \mathrm{d}z A(z)} \left[2A' + A^2 - 3B^2\right] &= k, \end{aligned} \tag{2.57}$$

In particular, we have that the two-manifold with metric $\mathrm{d}s^2 = e^{2\gamma}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$ has a constant scalar curvature. The general solution with $F = F(z)$ and $\check{\mathcal{R}}(\gamma)$ was studied in [4]: as one can verify, the complex variable F has the following form

$$F = \frac{2az + b}{az^2 + bz + c} \tag{2.58}$$

where a, b and c are complex integration constants (if these constants are real, we fall back in the "purely magnetic" case already considered in section 2.3). In [4] it was shown that setting $a \neq 0$ one recovers the supersymmetric Reissner-Nordström-Taub-NUT-AdS_4 (RNTN-AdS_4) solutions obtained in [18] (see eq. (B.1) for the relation with the NUT parameter and electric and magnetic charges). Let's consider now the case $a = 0$. We have

$$F = \frac{b}{bz + c} \tag{2.59}$$

with $b \neq 0$. Now we shift

$$z \to z - \frac{1}{2}\left(\frac{\bar{c}}{\bar{b}} + \frac{c}{b}\right) \tag{2.60}$$

and define the real constant

$$n \equiv \frac{i}{2}\left(\frac{\bar{c}}{\bar{b}} - \frac{c}{b}\right). \tag{2.61}$$

This yields

$$A = \frac{z}{z^2 + n^2}\,, \qquad B = \frac{n}{z^2 + n^2} \tag{2.62}$$

and, as a consequence, $k = -1$ and staticity (i.e. $\mathrm{d}\omega = 0$) for the four-dimensional solution. The solution reads

$$\mathrm{d}s^2 = -\frac{4(z^2 + n^2)}{\ell^2}\mathrm{d}t^2 + \frac{\ell^2}{4(z^2 + n^2)}\mathrm{d}z^2 + \frac{\ell^2}{2x^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{2.63}$$

$$\mathcal{F} = -\frac{\ell}{2x^2}\mathrm{d}x \wedge \mathrm{d}y \tag{2.64}$$

which is $\mathrm{AdS}_2 \times \mathbb{H}^2$ with magnetic flux on $\mathbb{H}^2$, where AdS_2 is written in global coordinates.

2.5 Harmonic solutions

Another rich class of solutions can be found if we choose F to be harmonic, $\Delta F = 0$. If we admit the possibility of isolated singularities to be present in the closure of the (x, y) domain, then this does not require (x, y)-independence. Eq. (2.2) gives

$$F = \frac{2az + b}{az^2 + bz + c}\,, \tag{2.65}$$

where a, b and c are complex functions of (x, y). Introducing the complex variable $\zeta = x+iy$, the harmonicity condition is equivalent to require that a, b and c are all holomorphic (or antiholomorphic) functions of ζ.

Next, from (2.4) one obtains

$$\phi = \ln|az^2 + bz + c| + \gamma(x, y)\,, \tag{2.66}$$

and (2.3) becomes

$$\check{\mathcal{R}}\,(\gamma) = 2\,(a\bar{c} + \bar{a}c) - b\bar{b}\,, \tag{2.67}$$

showing that the scalar curvature $\check{\mathcal{R}}(\gamma) = -\,e^{-2\gamma}\Delta 2\gamma$ of the two-dimensional metric $e^{2\gamma}(\mathrm{d}x^2 + \mathrm{d}y^2)$ is not constant in general.

For the shift vector we find

$$\begin{aligned} \omega_x - i\omega_y &= w_x - iw_y + \frac{\ell^2}{2}\int \partial_\zeta\left(F\bar{F}\right)\mathrm{d}z\,, && (2.68)\\ \partial_z(w_x - iw_y) &= 0\,, && (2.69)\\ \mathrm{d}w &= -i\frac{\ell^2}{2}e^{2\gamma}\left(a\bar{b} - b\bar{a}\right)\mathrm{d}x\wedge\mathrm{d}y\,. && (2.70) \end{aligned}$$

Here, $\mathrm{d} = \mathrm{d}x^i\partial_i$ denotes the exterior derivative in two dimensions. It follows that the form $w = w_i\mathrm{d}x^i$ can be expressed as $w = \check{\mathrm{d}}\psi$, where $\check{\mathrm{d}} \equiv \mathrm{d}x^i\epsilon_{ij}\partial_j$, and $\psi(x, y)$ is a function satisfying

$$\Delta\psi = i\frac{\ell^2}{2}e^{2\gamma}(a\bar{b} - b\bar{a})\,. \tag{2.71}$$

Note that, if we add a harmonic function ψ_0 to ψ, eq. (2.71) is still solved and the shift vector is just translated by w_0, where $w_0 = \check{\mathrm{d}}\psi_0$ is a closed form, $\mathrm{d}w_0 = 0$. Therefore, at least locally, $w_0 = \mathrm{d}v$ for some function v and the effect of ψ_0 can be reabsorbed in a diffeomorphism $u \mapsto u + v$. In other words, only solutions ψ of eq. (2.71) belonging to distinct cohomology classes produce physically different solutions.

Let us define the Eddington-Finkelstein-like coordinate $u = t + \frac{\ell^2}{4}\int F\bar{F}\mathrm{d}z$. Then, the metric takes the form

$$\mathrm{d}s^2 = -\frac{4}{\ell^2 F\bar{F}}\left(\mathrm{d}u + w\right)^2 + 2\left(\mathrm{d}u + w\right)\mathrm{d}z + \frac{\ell^2}{4}|2az + b|^2 e^{2\gamma}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{2.72}$$

and for the electromagnetic field one finds

$$\mathcal{F} = -\frac{i}{\ell}(\mathrm{d}u + w)\wedge\mathrm{d}\left(\frac{1}{F} - \frac{1}{\bar{F}}\right) - \frac{\ell}{4}|2az + b|^2\partial_z\left(\frac{1}{F} + \frac{1}{\bar{F}} - z\right)e^{2\gamma}\mathrm{d}x\wedge\mathrm{d}y\,, \tag{2.73}$$

which has the following potential

$$\mathcal{A} = \frac{i}{\ell}\left(\frac{1}{F} - \frac{1}{\bar{F}}\right)(\mathrm{d}u + w) + \frac{\ell}{2}\check{\mathrm{d}}\gamma\,. \tag{2.74}$$

Let us consider now some particular solutions. If the functions a, b and c are constant, we fall in the cases already studied in sections 2.3 and 2.4.3. More precisely, if $a = 0$ we obtain the anti-Nariai spacetime $\mathrm{AdS}_2 \times \mathbb{H}^2$, while for $a \neq 0$ the BPS limits of the RNTN-AdS_4 family of solutions are recovered. We will analyse now these two cases allowing for non-constant functions.

2.5.1 Supersymmetric Kundt solutions ($a = 0$)

When $a = 0$, eq. (2.71) tells us that ψ is an harmonic function and therefore, performing a diffeomorphism, we can take $w = 0$. Moreover, eq. (2.65) reads

$$F = \frac{b}{bz + c} \,. \tag{2.75}$$

Without loss of generality we can set $b = 1$ by rescaling accordingly the curvature of the transverse two-metric, and eq. (2.67) becomes $\check{\mathcal{R}}(\gamma) = -1$. Hence, the transverse two-metric has a constant negative curvature, and describes (at least locally) an hyperbolic plane $\mathbb{H}^2$. We can choose for example the solution $e^{2\gamma} = 2/x^2$. Finally, the metric and gauge field read

$$\mathrm{d}s^2 = -\frac{4}{\ell^2}|z + c|^2\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}z + \frac{\ell^2}{2x^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) , \tag{2.76}$$

$$\mathcal{A} = \frac{i}{\ell}\left(c - \bar{c}\right)\mathrm{d}u + \frac{\ell}{2x}\mathrm{d}y \,, \tag{2.77}$$

where c is an arbitrary holomorphic function $c = c(\zeta)$. This metric is precisely of the Kundt form, describing manifolds admitting a non-expanding, non-twisting null congruence of geodesics, and the solution has the interpretation of *supersymmetric electromagnetic and gravitational waves propagating on anti-Nariai spacetime*. This is a particular case of the more general solution found in [20]. Its Petrov type is II.

2.5.2 Supersymmetric Robinson-Trautman solutions ($a \neq 0$)

If, instead, $a \neq 0$, we can rewrite the function F in the following way

$$F = 2\frac{z + \beta}{(z + \beta)^2 - \delta} \,, \tag{2.78}$$

where $\beta \equiv b/2a$ and $\delta \equiv (b^2 - 4ac)/4a^2$ are two arbitrary holomorphic functions in ζ. The system of equations describing this class of supersymmetric configurations is

$$\phi = \ln\left|(z + \beta)^2 - \delta\right| + \gamma(x, y) \,, \tag{2.79}$$

$$\check{\mathcal{R}}(\gamma) = -4\left[2(\mathrm{Im}\,\beta)^2 + \mathrm{Re}\,\delta\right] , \tag{2.80}$$

$$\Delta\psi = 2\ell e^{2\gamma}\,\mathrm{Im}\,\beta \,. \tag{2.81}$$

In this case, the solution reads

$$\mathrm{d}s^2 = -\frac{\left|(z + \beta)^2 - \delta\right|^2}{\ell^2|z + \beta|^2}\left(\mathrm{d}u + w\right)^2 + 2\left(\mathrm{d}u + w\right)\mathrm{d}z + \ell^2|z + \beta|^2e^{2\gamma}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) , \tag{2.82}$$

with the following electromagnetic potential

$$\mathcal{A} = -\frac{\mathrm{Im}\left[\left((z + \beta)^2 - \delta\right)\left(z + \bar{\beta}\right)\right]}{\ell\,|z + \beta|^2}\left(\mathrm{d}u + w\right) + \frac{\ell}{2}\check{\mathrm{d}}\gamma \,. \tag{2.83}$$

If β and δ are constant functions, then, as already stressed, this solution is of Petrov type D and belongs to the RNTN-AdS$_4$ family of solutions. Allowing for a ζ-dependence in these functions deforms the metric, which acquire a non-vanishing Weyl scalar Ψ_4, signaling the presence of gravitational radiation. In general, solution (2.83) describes electromagnetic and gravitational expanding waves propagating on a supersymmetric RNTN-AdS$_4$ background. To our knowledge, these field configurations where not known previously in the literature.

We shall work out in the following the simplest of these solutions, leaving the general analysis for further investigations. Suppose that $\operatorname{Im}\beta = 0$ (this condition corresponds to put the NUT parameter n of the RNTN-AdS$_4$ solution to zero in the case of constant a, b and c). Since β is holomorphic, this implies that β should be a real constant, hereafter named κ. It follows from eq. (2.81) that ψ is harmonic and therefore we can take $w = 0$. The resulting spacetime and gauge field are

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{\left|(z+\kappa)^2-\delta\right|^2}{\ell^2(z+\kappa)^2}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}z + \ell^2(z+\kappa)^2 e^{2\gamma}\left(\mathrm{d}x^2+\mathrm{d}y^2\right), \\ \mathcal{A} &= \frac{\operatorname{Im}\delta}{\ell\,(z+\kappa)}\mathrm{d}u + \frac{\ell}{2}\check{\mathrm{d}}\gamma\,, \end{aligned} \tag{2.84}$$

with $\check{\mathcal{R}}(\gamma) = -4\operatorname{Re}\delta$. The constant κ can then be reabsorbed by the shift $z \to z-\kappa$ and the solution becomes

$$\mathrm{d}s^2 = -\left|\frac{z}{\ell} - \frac{\delta}{\ell z}\right|^2 \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}z + \ell^2 z^2 e^{2\gamma}\left(\mathrm{d}x^2+\mathrm{d}y^2\right), \tag{2.85}$$

$$\mathcal{A} = \frac{\operatorname{Im}\delta}{\ell z}\,\mathrm{d}u + \frac{\ell}{2}\check{\mathrm{d}}\gamma\,. \tag{2.86}$$

This metric is clearly of the Robinson-Trautman form, describing manifolds admitting an expanding, non-twisting null congruence of geodesics. To put it in a more familiar shape, we can define the function $P = \sqrt{2}\ell^{-1}e^{-\gamma}$ and the operator $\Delta^* \equiv \frac{1}{2}P^2\Delta$, so that $\check{\mathcal{R}}(\gamma) = 2\ell^2\Delta^*\ln P$ and the solution reads

$$\mathrm{d}s^2 = -\left[\frac{z^2}{\ell^2} + \Delta^*\ln P + \frac{|\delta|^2}{\ell^2 z^2}\right]\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}z + \frac{2z^2}{P^2}\left(\mathrm{d}x^2+\mathrm{d}y^2\right), \tag{2.87}$$

$$\mathcal{A} = \frac{\operatorname{Im}\delta}{\ell z}\,\mathrm{d}u - \frac{\ell}{2}\check{\mathrm{d}}\ln P\,. \tag{2.88}$$

with $P(x,y)$ any solution of $\Delta^*\Delta^*\ln P = 0$, while $\operatorname{Im}\delta$ is determined by the fact that δ is holomorphic and its real part is fixed by $\check{\mathcal{R}}(\gamma)$. This solution is of Petrov type-II and generalizes the massless and purely gravitational Robinson-Trautman class of solutions found in [20], by adding electromagnetic waves on it. In conclusion, we can interpret the solution (2.85), (2.86) as *supersymmetric electromagnetic and gravitational expanding waves propagating on the BPS Reissner-Nordström-AdS$_4$ spacetimes.*

JHEP07(2004)061

3. The lightlike case

In [4] it was shown that the general supersymmetric solution in the lightlike case is an electrovac travelling wave with metric[8]

$$\mathrm{d}s^2 = \frac{\ell^2}{x^2}\left[\mathcal{G}(x,y,u)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2\right], \tag{3.1}$$

and the null electromagnetic field is given by

$$\mathcal{F} = \mathrm{d}\mathcal{A} = \varphi'(u)\,\mathrm{d}u \wedge \mathrm{d}x\,, \qquad \mathcal{A} = \varphi(u)\,\mathrm{d}x\,. \tag{3.2}$$

Here, the arbitrary function $\varphi'(u)$ defines the profile of the electromagnetic wave propagating on this metric, while $\mathcal{G}(x,y,u)$ is any solution of the inhomogeneous Siklos equation [21]

$$\Delta\mathcal{G} - \frac{2}{x}\partial_x\mathcal{G} = -\frac{4x^2}{\ell^2}\left(\varphi'\right)^2. \tag{3.3}$$

The dependence of $\mathcal{G}(x,y,u)$ on u describes the profile of the gravitational wave. The general solution to this equation was obtained in [21], and reads (cf. also appendix C)

$$\mathcal{G}(\zeta,\bar\zeta,u) = \frac{1}{2}\left(\zeta+\bar\zeta\right)\left(\partial f + \bar\partial\bar f\right) - \left(f+\bar f\right) - \frac{\left(\varphi'(u)\right)^2}{16\ell^2}\left(\zeta+\bar\zeta\right)^4, \tag{3.4}$$

where $f(\zeta,u)$ is an arbitrary holomorphic function in $\zeta = x+iy$.

This family of travelling waves enjoys a large group of coordinate transformations which preserve the form (3.1) of the line element. Under the diffeomorphism $(u,v,x,y)\mapsto(\bar u,\bar v,\bar x,\bar y)$ defined by

$$\begin{aligned}\bar u &= \chi(u)\,, \qquad \bar x = x\sqrt{\chi'(u)}\,, \qquad \bar y = y\sqrt{\chi'(u)} - \psi(u)\,,\\ \bar v &= v - \frac{\chi''(u)}{4\chi'(u)}\left(x^2+y^2\right) + \frac{\psi'(u)}{\sqrt{\chi'(u)}}y + \gamma(u)\,,\end{aligned} \tag{3.5}$$

where $\chi(u)$, $\psi(u)$ and $\gamma(u)$ are arbitrary functions of u, the metric keeps the same form (3.1) but with $\bar{\mathcal{G}}(\bar x,\bar y,\bar u)$ given by[9]

$$\bar{\mathcal{G}}(\bar x,\bar y,\bar u) = \frac{1}{\chi'(u)}\left[\mathcal{G}(x,y,u) + \frac{1}{2}\{\chi(u);u\}(x^2+y^2) - 2\gamma'(u)\right] - \frac{2y}{\sqrt{\chi'(u)}}\left(\frac{\psi'(u)}{\chi'(u)}\right)' - \left(\frac{\psi'(u)}{\chi'(u)}\right)^2. \tag{3.6}$$

Here, the prime denotes the derivative with respect to u, while

$$\{\chi(u);u\} = \frac{\chi'''(u)}{\chi'(u)} - \frac{3}{2}\left(\frac{\chi''(u)}{\chi'(u)}\right)^2 \tag{3.7}$$

defines the Schwarzian derivative. These are not Killing symmetries, the metric changes, but solutions are brought into other solutions. The special diffeomorphisms with $\psi(u) =$

[8] For nonabelian generalizations see [27].

[9] This invariance was found by Siklos [21]. Here we use $\chi'(u) = e^{-2\phi(u)}$, and (3.6) corrects a typeset error in equation (34) of [21].

$\gamma(u) = 0$ correspond to reparameterizations of the coordinate u; this transformation group is generated by a central extension of the Virasoro algebra [22].

All metrics (3.1) where shown to preserve at least one quarter of the supersymmetries in [4]. Indeed, it is easy to show that the spinor

$$\epsilon = \frac{1}{4}\Gamma_-\Gamma_+ \left(1 + \Gamma_x\right) e^{\frac{ix}{\ell}\varphi(u)}\epsilon_0 \tag{3.8}$$

solves the Killing spinor equation for all these configurations, and therefore the generalized Siklos spacetimes (3.1) preserve at least one supersymmetry. Here, $\frac{1}{4}\Gamma_+\Gamma_-(1+\Gamma_x)$ is a projection operator of rank one[10] and $\Gamma_+\Gamma_-$ is just the usual chirality projector appearing in the Killing spinors of supersymmetric pp-waves. The aim of this section is to solve the Killing spinor equation for these backgrounds, in order to obtain the exact fraction of supersymmetry preserved, and thus to get a complete classification of supersymmetric solutions in the lightlike case.

3.1 First integrability conditions

As a first step, we shall solve the (first) integrability conditions $[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0$. Although these conditions are only necessary [23], they impose some algebraic conditions on the functions $\mathcal{G}$ and φ, and simplify the task of solving the Killing spinor equations.

The vanishing of the supercurvature yields two nontrivial constraints,

$$\varphi'(u)\Gamma_- \left(1 - \Gamma_x\right)\epsilon = 0\,, \tag{3.9}$$

and

$$\frac{x}{2\ell}\left(\frac{2ix}{\ell}\varphi''(u) + \frac{1}{2}\Delta_-\mathcal{G} + \mathcal{G}_{,xy}\Gamma_y\right)\Gamma_-\epsilon - \frac{3i}{\ell}\varphi'(u)\left(1 - \Gamma_x\right)\epsilon = 0\,, \tag{3.10}$$

with Δ_- defined by $\Delta_- = \partial_x^2 - \partial_y^2$.

Using the representation of the Dirac matrices given in appendix A, which is well-adapted to the present problem, these relations become respectively

$$\varphi'(u)\epsilon_2 = 0\,, \tag{3.11}$$

and

$$\begin{pmatrix} 0 & \frac{x}{\sqrt{2}\ell}\left(\frac{2ix}{\ell}\varphi''(u) + \frac{1}{2}\Delta_-\mathcal{G}\right) & \frac{x}{\sqrt{2}\ell}\mathcal{G}_{,xy} & 0 \\ 0 & \frac{6i}{\ell}\varphi'(u) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{x}{\sqrt{2}\ell}\mathcal{G}_{,xy} & \frac{x}{\sqrt{2}\ell}\left(\frac{2ix}{\ell}\varphi''(u) - \frac{1}{2}\Delta_-\mathcal{G}\right) & \frac{6i}{\ell}\varphi'(u) \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = 0\,, \tag{3.12}$$

where ϵ_i are the spinor components. The first relation suggest to study two distinct cases, the purely gravitational one $\varphi'(u) = 0$ and the case where electromagnetic fields are present.

[10] The Dirac matrices $\Gamma_\pm$ and Γ_x are defined in appendix A.

3.2 Purely gravitational waves

This case has been extensively studied in [21] (see [25] for a generalization to higher dimensions). When $\varphi'(u) = 0$, only equation (3.12) yields new conditions and the rank of this matrix tells us the number of Killing spinors eliminated by the integrability conditions. There are two cases to be analysed. First, if $(\Delta_{-}\mathcal{G})^2 + 4\left(\mathcal{G}_{,xy}\right)^2 = 0$ the matrix vanishes. Otherwise its rank is two.

3.2.1 The maximally supersymmetric case: $\Delta_{-}\mathcal{G} = \mathcal{G}_{,xy} = 0$

These conditions are satisfied if and only if the spacetime is AdS [21]. It follows that the most general way of expressing AdS in the form (3.1) corresponds to $f(\zeta, u)$ at most quadratic in ζ, and it can be generated by a coordinate transformation (3.5) from the metric with $\mathcal{G} = 0$. In this case the matrix in equation (3.12) vanishes and no condition is imposed by (3.11) and (3.12). In fact, it is well-known that AdS is a maximally supersymmetric spacetime [26] and recently it has been shown that this is the only maximally supersymmetric solution of the model under consideration [4]. For the sake of completeness, we give here the four independent Killing spinors, which in our conventions read

$$
\begin{aligned}
\epsilon^{(1)} &= (1,0,0,0)\,, & \epsilon^{(2)} &= \left(\sqrt{2}\frac{v}{\ell},1,\frac{y}{x},0\right), \\
\epsilon^{(3)} &= \left(0,0,\frac{1}{x},0\right), & \epsilon^{(4)} &= (-y,0,0,x).
\end{aligned}
\tag{3.13}
$$

3.2.2 One quarter BPS Lobatchevski waves: $\Delta_{-}\mathcal{G} \neq 0$ or $\mathcal{G}_{,xy} \neq 0$

In this case $f(\zeta, u)$ is at least of order three in ζ and the spacetime describes an exact AdS gravitational wave [24]. The matrix (3.12) has rank two and the configuration is at most one half BPS. However, while solving the Killing spinor equation, an additional condition on the Killing spinor emerges, leaving just one quarter of the supersymmetries, as shown in [25]. The residual supersymmetry is generated by the Killing spinor (3.8). This is a very simple example that shows that the vanishing of the supercurvature is not a sufficient condition to ensure the existence of Killing spinors.

3.3 The electromagnetic case

Let us now turn on the electromagnetic field and consider the case where $\varphi'(u) \neq 0$. The first integrability condition (3.11) imposes now $\epsilon_2 = 0$. Then, the second integrability condition (3.12) simplifies to the following two relations,

$$
\mathcal{G}_{,xy}\epsilon_3 = 0\,, \qquad \frac{x}{\sqrt{2}\ell}\left(\frac{2ix}{\ell}\varphi''(u) - \frac{1}{2}\Delta_{-}\mathcal{G}\right)\epsilon_3 + \frac{6i}{\ell}\varphi'(u)\epsilon_4 = 0\,. \tag{3.14}
$$

The former relation suggests us to examine separately the cases $\mathcal{G}_{,xy} \neq 0$ and $\mathcal{G}_{,xy} = 0$.

3.3.1 Generic one quarter BPS solution: $\varphi'(u) \neq 0$ and $\mathcal{G}_{,xy} \neq 0$

In this case, we have $\epsilon_3 = \epsilon_4 = 0$ in addition to $\epsilon_2 = 0$ and the only Killing spinor available is (3.8). Hence, these spacetimes are exactly one quarter supersymmetric.

JHEP07(2004)061

3.3.2 One half BPS waves: $\varphi'(u) \neq 0$ and $\mathcal{G}_{,xy} = 0$

The most general solution of the Siklos equation restricted by these conditions is given by

$$\mathcal{G}(x,y,u) = -\frac{1}{\ell^2}\varphi'(u)^2 x^4 + \frac{1}{6}\xi_3(u)x^3 + \frac{1}{2}\xi_2(u)\left(x^2+y^2\right) + \xi_1(u)y + \xi_0(u)\,, \qquad (3.15)$$

where $\xi_i(u)$ are arbitrary real functions of u. Then, $\Delta_-\mathcal{G} = \xi_3(u)x - \frac{12}{\ell^2}\varphi'(u)^2x^2$ and (3.12) becomes a relation between the components ϵ_3 and ϵ_4 of the Killing spinor,

$$\frac{x}{\sqrt{2}}\left(\frac{2ix}{\ell}\varphi''(u) + \frac{6x^2}{\ell^2}\varphi'^2(u) - \frac{x}{2}\xi_3(u)\right)\epsilon_3 + 6i\varphi'(u)\epsilon_4 = 0\,. \qquad (3.16)$$

As we have also $\epsilon_2 = 0$, in this case there can be at most two independent components of the Killing spinor. Let us solve now the Killing spinor equations to see whether the existence of the second supersymmetry imposes further constraints. Some lenghty but straightforward algebra shows that these equations are solved by

$$\epsilon_1 = \frac{1}{\sqrt{2}\ell}\left(A + B(u) - y\kappa'(u)\right)e^{\frac{ix}{\ell}\varphi(u)} \qquad \epsilon_2 = 0\,, \qquad (3.17)$$

$$\epsilon_3 = \frac{\kappa(u)}{x}e^{\frac{ix}{\ell}\varphi(u)}\,, \qquad \epsilon_4 = \frac{x}{\sqrt{2}\ell}\left(\kappa'(u) + \frac{ix}{\ell}\kappa(u)\varphi'(u)\right)e^{\frac{ix}{\ell}\varphi(u)}\,, \qquad (3.18)$$

where A is an arbitrary constant while $B(u)$ and $\kappa(u)$ are two complex functions subject to the conditions

$$B'(u) + \frac{1}{2}\xi_1(u)\kappa(u) = 0\,, \qquad (3.19)$$

$$\kappa''(u) - \frac{1}{2}\xi_2(u)\kappa(u) = 0\,. \qquad (3.20)$$

Now the relation (3.16) can be solved for $\kappa(u)$,

$$\kappa(u) = \kappa_0\left(\varphi'(u)\right)^{-\frac{1}{3}}\exp\left(-\frac{i\ell}{12}\int\frac{\xi_3(u)}{\varphi'(u)}\mathrm{d}u\right)\,, \qquad (3.21)$$

with κ_0 an arbitrary complex constant, and we are left with the additional consistency condition (3.20). Since the ξ_i's are real, the real and imaginary parts of the constraint determine the functions $\xi_2(u)$ and $\xi_3(u)$ to be

$$\begin{aligned}\xi_2(u) &= \frac{8}{9}\left(\frac{\varphi''(u)}{\varphi'(u)}\right)^2 - \frac{2}{3}\frac{\varphi'''(u)}{\varphi'(u)} - 2\alpha^2\ell^2\left[\varphi'(u)\right]^{4/3}\\ \xi_3(u) &= 12\alpha\left[\varphi'(u)\right]^{5/3}\,.\end{aligned} \qquad (3.22)$$

whith α an arbitrary real integration constant. In conclusion, if $\mathcal{G}(x,y,u)$ has the form (3.15), with $\xi_2(u)$ and $\xi_3(u)$ given by (3.22), the solution preserves exacly one half of the supersymmetries, otherwise it is only one quarter BPS. The family of half BPS solutions is parameterized by two real functions $\xi_0(u)$, $\xi_1(u)$ and a real number α. Howerver, many of these solutions are related by diffeomorphism, and are in fact equivalent.

JHEP07(2004)061

To understand better the nature of these one half supersymmetric solutions, we can put them in a canonical form by performing a suitable change of coordinates. In particular, we choose

$$\chi(u) = \int \left(\varphi'(u)\right)^{\frac{2}{3}} \mathrm{d}u\,, \tag{3.23}$$

$$\psi'(u) = \frac{1}{2}\left(\varphi'(u)\right)^{\frac{2}{3}} \int \frac{\xi_1(u)}{\left(\varphi'(u)\right)^{1/3}}\mathrm{d}u\,, \tag{3.24}$$

$$\gamma'(u) = \frac{1}{2}\xi_0(u) - \frac{\left(\psi'(u)\right)^2}{2\left(\varphi'(u)\right)^{2/3}}\,. \tag{3.25}$$

The effect of $\chi(u)$ is to bring the field strength in the form $\mathcal{F} = \mathrm{d}\bar{u}\wedge\mathrm{d}\bar{x}$ and to simplify $\xi_2(u)$; the functions $\psi(u)$ and $\gamma(u)$ eliminate the linear term in y and the $\xi_0(u)$ term in $\mathcal{G}(x, y, u)$. Finally, the general one half BPS solution is, up to diffeomorphisms, given by

$$\mathcal{G}_\alpha(x, y, u) = -\frac{x^4}{\ell^2} + 2\alpha x^3 - \alpha^2\ell^2\left(x^2+y^2\right), \qquad \varphi(u) = u\,, \tag{3.26}$$

and parameterized by a single real number α. The nonvanishing components of the corresponding Killing spinors are

$$\begin{aligned}
\epsilon_1 &= \frac{A}{\sqrt{2}\ell}\exp\left[\frac{iux}{\ell}\right] + \frac{i\kappa_0\alpha}{\sqrt{2}}y\exp\left[\frac{iu}{\ell}\left(x-\alpha\ell^2\right)\right],\\
\epsilon_3 &= \frac{\kappa_0}{x}\exp\left[\frac{iu}{\ell}\left(x-\alpha\ell^2\right)\right],\\
\epsilon_4 &= \frac{i\kappa_0 x}{\sqrt{2}\ell^2}\left(x-\alpha\ell^2\right)\exp\left[\frac{iu}{\ell}\left(x-\alpha\ell^2\right)\right].
\end{aligned} \tag{3.27}$$

The Killing spinor (3.8), common to all lightlike solutions, is recovered by setting $\kappa_0 = 0$ and $A = 1$, while the other independent Killing spinor is obtained by taking $\kappa_0 = 1$ and $A = 0$.

It is finally important to distinguish between the $\alpha = 0$ and $\alpha \neq 0$ solutions. Siklos has classified the spacetimes of the form (3.1) according to the number of independent Killing vectors [21]. It follows that if $\alpha = 0$ the spacetime admits a five-dimensional group of isometries, generated by five Killing vectors. In fact, this solution is exactly the solution (2.28) of the timelike case, and the groups of isometries agree.

On the other hand, when $\alpha \neq 0$, the canonical form of $\mathcal{G}$ falls in the $A(x, y)$ class of [21], meaning that we have just the two trivial Killing vectors ∂_u and ∂_v.

In table 1 we summarize the complete classification of supersymmetric solutions of the lightlike case.

3.4 One half BPS lightlike solutions

We will now analyze more in detail the one half BPS solutions with $\mathcal{G} = \mathcal{G}_\alpha$, (3.26). In particular we can compute the norm squared of the Killing vectors constructed from the Killing spinors, in order to see if they also belong to the timelike class of the theory. Using

JHEP07(2004)061

Lightlike case	purely gravitational solutions $\varphi'(u) = 0$	purely gravitational solutions $\varphi'(u) = 0$
One quarter BPS	Lobatchevski wave	$\mathcal{G} \neq \mathcal{G}_\alpha$
One half BPS	*none*	$\mathcal{G}_\alpha(u, x, y)$ (3.26)
Three quarters BPS	*none*	*none*
Maximally SUSY	AdS_4	*none*

Table 1: Classification of supersymmetric spacetimes in the lightlike case. Note that the fraction 3/4 of supersymmetry cannot be preserved.

the definition $V_\mu = i\bar{\epsilon}\Gamma_\mu\epsilon$ and choosing the Γ-matrix representation given in appendix A, we obtain for the components of V in the vierbein frame (see appendix D),

$$\begin{aligned} V_+ &= -\frac{1}{\sqrt{2}\ell^4}\left[|k_0|^2 x^2 \left(x - \alpha\ell^2\right)^2 + \ell^2 \left|A + ik_0\alpha\ell y e^{-iu\alpha\ell}\right|^2\right], \\ V_- &= \frac{\sqrt{2}|k_0|^2}{x^2}, \\ V_x &= 0, \\ V_y &= -\frac{\sqrt{2}}{\ell x}\,\mathrm{Re}\left(A\bar{k}_0 e^{iu\alpha\ell}\right). \end{aligned}$$

The norm squared of V is given by

$$V^2 = -\frac{2|k_0|^2}{\ell^4 x^2}\left[|k_0|^2 x^2 \left(x - \alpha\ell^2\right)^2 + \ell^2 \left|A + ik_0\alpha\ell y e^{-iu\alpha\ell}\right|^2\right] + \frac{2}{\ell^2 x^2}\,\mathrm{Re}\left(A\bar{k}_0 e^{iu\alpha\ell}\right)^2.$$

If $k_0 = 0$ we have $V^2 = 0$, so in this case the spinor generates the null vector typical of the lightlike solutions. Instead, if $A = 0$ one obtains

$$V^2 = -\frac{2|k_0|^4}{\ell^4 x^2}\left[x^2\left(x - \alpha\ell^2\right)^2 + \alpha^2\ell^4 y^2\right] = \frac{2|k_0|^4}{\ell^2 x^2}\mathcal{G}_\alpha\,, \tag{3.28}$$

which is negative, so the solution also belongs to the timelike class. Therefore every one half supersymmetric lightlike solution is also a timelike solution. In order to write down this geometry in the form (2.1), we also need the other tensors $f = \bar{\epsilon}\epsilon$, $g = i\bar{\epsilon}\Gamma_5\epsilon$ and $A_\mu = i\bar{\epsilon}\Gamma_5\Gamma_\mu\epsilon$ used in [4]. A little bit of algebra yields

$$\begin{aligned} f &= \frac{\sqrt{2}|k_0|^2}{\ell^2}\left(x - \alpha\ell^2\right), \\ g &= -\sqrt{2}|k_0|^2\alpha\frac{y}{x}, \\ F &= \frac{2i}{\ell}\frac{1}{f - ig} = i\frac{\sqrt{2}\ell}{|k_0|^2}\frac{1}{(x - \alpha\ell^2) + i\alpha\ell^2\frac{y}{x}}, \\ A_\mu \mathrm{d}x^\mu &= -\frac{\sqrt{2}|k_0|^2}{\ell}\mathrm{d}\left[\frac{y}{x}\left(x - \alpha\ell^2\right)\right]. \end{aligned}$$

For simplicity we set $\sqrt{2}|k_0|^2/\ell = 1$, so that one has $V^2 = \mathcal{G}_\alpha/x^2$, and $V = -\ell^{-1}\partial_u$. Comparing this with $V = \partial_t$ [4], we can then identify $t \equiv -\ell u$ and $\omega \equiv -\ell\mathcal{G}_\alpha^{-1}\mathrm{d}v$. Furthermore,

$A_\mu \mathrm{d}x^\mu = \mathrm{d}z$ [4] implies $z = -\frac{y}{x}\left(x - \alpha\ell^2\right)$. The diffeomorphism

$$Y = -\frac{z^2\alpha\ell x^2}{2(x-\alpha\ell^2)^2} + \frac{x^3}{3\ell} - \frac{\alpha\ell x^2}{2} + C\,, \qquad X = v\,, \tag{3.29}$$

where C denotes an arbitrary constant, brings the metric finally into the form (2.1), with coordinates z, X, Y, and with F, ϕ and ω given by

$$F = \frac{2i(x-\alpha\ell^2)}{(x-\alpha\ell^2)^2 - i\alpha\ell^2 z}\,, \qquad e^{2\phi} = \frac{\ell^2}{x^4}\,, \qquad \omega = \frac{\ell^3 F\bar{F}}{4x^2}\mathrm{d}X\,. \tag{3.30}$$

In (3.30), x is to be understood as a function of Y and z defined by (3.29). We have checked explicitely that F, ϕ and ω satisfy the equations (2.2)–(2.5). The electromagnetic field strength reads $\mathcal{F} = \ell^{-1}\mathrm{d}x \wedge \mathrm{d}t$.

3.5 Lifting to eleven dimensions

We can lift the general lightlike geometry (3.1) to a solution of eleven-dimensional supergravity using the results of [28, 29]. The reduction ansatz for the metric reads

$$\mathrm{d}s_{11}^2 = \mathrm{d}s_4^2 + 4\ell^2 \sum_{i=1}^4 \left[\mathrm{d}\mu_i^2 + \mu_i^2\left(\mathrm{d}\phi_i + \frac{1}{2\ell}\mathcal{A}\right)^2\right]\,, \tag{3.31}$$

where the round metric on the seven-sphere is written as

$$\mathrm{d}\Omega_7^2 = \sum_{i=1}^4 \left(\mathrm{d}\mu_i^2 + \mu_i^2 \mathrm{d}\phi_i^2\right)\,.$$

The μ_i, which satisfy $\sum_i \mu_i^2 = 1$, can be parametrized in terms of angles on the three-sphere as

$$\mu_1 = \sin\theta\,, \qquad \mu_2 = \cos\theta\sin\phi\,, \qquad \mu_3 = \cos\theta\cos\phi\sin\psi\,, \qquad \mu_4 = \cos\theta\cos\phi\cos\psi\,.$$

The reduction ansatz for the four-form field strength is given by

$$\mathcal{F}_{[4]} = -\frac{3}{\ell}\epsilon_{[4]} - 2\ell^2 \sum_{i=1}^4 d(\mu_i^2) \wedge d\phi_i \wedge {}^*\mathcal{F}\,, \tag{3.32}$$

with $*$ denoting the Hodge dual with respect to the four-dimensional metric and $\epsilon_{[4]}$ its volume form.

In our case we find it convenient to choose $\mathcal{A} = -x\varphi'(u)\,\mathrm{d}u$. The dual of the electromagnetic field strength reads

$${}^*\mathcal{F} = \varphi'(u)\mathrm{d}u \wedge \mathrm{d}y\,.$$

This yields finally for the eleven-dimensional metric, the four-form field strength and the three-form gauge potential respectively

$$\mathrm{d}s_{11}^2 = \frac{\ell^2}{x^2}\left[\mathcal{G}(x,y,u)\,\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2\right] +$$

$$
\begin{aligned}
&+4\ell^2 \sum_i \left[\mathrm{d}\mu_i^2 + \mu_i^2 \left(\mathrm{d}\phi_i - \frac{1}{2\ell} x\varphi'(u)\mathrm{d}u \right)^2 \right], \\
\mathcal{F}_{[4]} &= -\frac{3}{\ell}\epsilon_{[4]} + 2\ell^2 \varphi'(u) \sum_i \mathrm{d}(\mu_i^2) \wedge \mathrm{d}\phi_i \wedge \mathrm{d}u \wedge \mathrm{d}y\,, \\
\mathcal{A}_{[3]} &= \frac{\ell^3}{x^3}\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}y + 2\ell^2 \varphi'(u) \sum_i \mu_i^2 \mathrm{d}\phi_i \wedge \mathrm{d}u \wedge \mathrm{d}y\,.
\end{aligned} \tag{3.33}
$$

A special case appears for $\mathcal{G}(x,y,u) = -x^4/\ell^2$ (which is obtained for $\alpha = 0$ from the half supersymmetric solutions (3.26)). The metric in eleven dimensions is then given by

$$
\mathrm{d}s_{11}^2 = \frac{\ell^2}{x^2}\left[2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2\right] + 4\ell^2 \mathrm{d}\Omega_7^2 - 4\ell x \sum_i \mu_i^2 \mathrm{d}\phi_i\, \mathrm{d}u\,,
$$

which differs from $\mathrm{AdS}_4 \times \mathrm{S}^7$ only by the last term, which describes rotation along the S^7.

It would be interesting to see exactly how many of the 32 supercharges are preserved by the solution (3.33). (We know that they preserve at least two real supercharges if $\mathcal{G} \neq \mathcal{G}_\alpha$ and four if $\mathcal{G} = \mathcal{G}_\alpha$, but there might be more). Furthermore, the form of (3.33) suggests that the eleven-dimensional solutions might have an interpretation as the near-horizon limit of rotating M2-branes with a gravitational wave along the brane. We will leave these points for future investigations.

4. Final remarks

In section 3 a complete classification of the lightlike case was obtained: we showed that the solutions preserve either one quarter or one half of the supersymmetry, and that a fraction of three quarters is not possible. The explicit form of the wave profile for a geometry with two supercovariantly constant spinors was given. In this case, it was shown that the second Killing spinor gives rise to a timelike Killing vector. It would be nice to have a complete classification of the timelike case as well. For instance, it might be that BPS solutions with three supercovariantly constant spinors exist in that subclass. In order to obtain such a classification, one should plug the general form (2.1) into the integrability conditions $[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon = 0$, and determine the rank of these matrices. Unfortunately, the timelike solution (2.1) is much less explicit than the lightlike one, eq. (3.1), so technically this task is not so easy.

We saw in section 2 that a kind of "dimensional reduction" (i. e. , z-independence) of the general equations (2.2)–(2.4) yields the dimensionally reduced gravitational Chern-Simons action. This leads us to ask the question if the complete system (2.2)–(2.4) is described by the gravitational Chern-Simons theory in three dimensions [30]. It would be interesting to pursue this point further, since a complete understanding of the mathematical structure behind this set of differential equations would simplify significantly the explicit construction of all supersymmetric solutions.

Finally, there is an interesting aspect related to the $\mathrm{PSL}(2,\mathbb{R})$ transformations (2.6), (2.7). Some months ago, Witten showed that there is a natural action of the group $\mathrm{SL}(2,\mathbb{Z})$ on the space of three-dimensional conformal field theories with U(1) symmetry and a chosen coupling to a background gauge field [31]. He argued that for CFTs with AdS_4 dual, the $\mathrm{SL}(2,\mathbb{Z})$ action on the three-dimensional CFT may be viewed as the holographic image of the well-known $\mathrm{SL}(2,\mathbb{Z})$ duality of electrodynamics on AdS_4. More recently, Leigh and Petkou showed that the group $\mathrm{SL}(2,\mathbb{Z})$ acts also on the two-point functions of the energy-momentum tensor of three-dimensional CFTs, and suggested that the holographic image of this action (if any) should be an appropriate generalization of electromagnetic duality invariance to include also gravity. Now, looking at (B.2), we see that our $\mathrm{SL}(2,\mathbb{R})$ transformations mix electric and magnetic charges, but they act also on the gravitational field. They might thus (in the case of supersymmetry) be a candidate for the mentioned generalization of electromagnetic duality at the nonlinear level, which includes also gravity. A possible check of the relevance of (2.6), (2.7) in this context would be to do a Fefferman-Graham expansion of the electromagnetic field strength $\mathcal{F}$ near the boundary, and see if the $\mathrm{PSL}(2,\mathbb{R})$ transformations exchange the two different possible boundary conditions.

Acknowledgments

This work was partially supported by INFN, MURST and by the European Commission RTN program HPRN-CT-2000-00131, in which S. L. C. , M. M. C. and D. K. are associated to the University of Torino. We are grateful to L. Vanzo for useful discussions.

A. Conventions

Throughout this paper, the conventions are as follows: $a,b,\ldots$ refer to $D=4$ tangent space indices, and $\mu,\nu,\ldots$ refer to $D=4$ world indices. The signature is $(-,+,+,+)$, $\varepsilon_{0123}=+1$.

The gamma matrices are defined to satisfy the four-dimensional Clifford algebra $\{\Gamma_a,\Gamma_b\}=2\eta_{ab}$, and the parity matrix is $\Gamma_5=i\Gamma_{0123}$. We antisymmetrize with unit weight, i. e. $\Gamma_{ab}\equiv\Gamma_{[a}\Gamma_{b]}\equiv\frac{1}{2}[\Gamma_a,\Gamma_b]$ etc. The Dirac conjugate is defined by $\bar{\psi}=i\psi^\dagger\Gamma^0$.

Late latin indices $i,j,\ldots$ refer to two-dimensional submanifolds. They can take the values 1, 2, and $\varepsilon_{12}=+1$.

In section 3, we use a prime to denote differentiation with respect to u, the complex coordinate ζ is defined by $\zeta=x+iy$, and ∂, $\bar{\partial}$ indicate the derivation with respect to ζ and $\bar{\zeta}$ respectively. The differential operator Δ_- is defined by

$$\Delta_-=\partial_x^2-\partial_y^2=2\left(\partial^2+\bar{\partial}^2\right). \tag{A.1}$$

Finally, to write down the matrix form of the Killing spinor equations we use the following representation of the Dirac matrices,

$$\Gamma_0=\begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2\end{pmatrix}, \qquad \Gamma_1=\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}, \tag{A.2}$$

$$\Gamma_2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, \tag{A.3}$$

with

$$\Gamma_5 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \tag{A.4}$$

and $\epsilon_1, \ldots, \epsilon_4$ are the components of the spinor in this basis.

B. PSL$(2,\mathbb{R})$ transformations

In this section we show that the PSL$(2,\mathbb{R})$ transformations (2.6), (2.7) can be used to generate new solutions from known ones. As an example we use the supersymmetric Reissner-Nordström-Taub-NUT-AdS$_4$ solution [18], which is obtained for [4]

$$\begin{aligned} F &= \frac{2(z+in)}{(z+in)^2 - i\ell(Q-iP)}, \\ e^{2\phi} &= \frac{4[(z+in)^2 - i\ell(Q-iP)][(z-in)^2 + i\ell(Q+iP)]}{\ell^2[1+(x^2+y^2)^2]}, \end{aligned} \tag{B.1}$$

where n denotes the NUT parameter, and Q and P are the electric and magnetic charges respectively. The latter is subject to the charge quantization condition

$$-2\ell P = \ell^2 + 4n^2,$$

which comes from eq. (2.3). We see that $e^{2\phi}(dx^2+dy^2)$ represents the metric on the round two-sphere, and for simplicity we will consider this case only, although generalizations to hyperbolic or flat spaces exist [18, 4].

After applying a PSL$(2,\mathbb{R})$ transformation, F and ϕ take again the form (B.1), but with different parameters $\tilde{n}$, $\tilde{Q}$ and $\tilde{P}$ that are related to n, Q, P by

$$\begin{aligned} \tilde{n} &= |\lambda| \operatorname{Im}\alpha, \\ \ell\tilde{Q} &= |\lambda|^2 (2\operatorname{Re}\alpha \operatorname{Im}\alpha - \operatorname{Im}\beta), \\ \ell\tilde{P} &= |\lambda|^2 (\operatorname{Re}^2\alpha - \operatorname{Im}^2\alpha - \operatorname{Re}\beta), \end{aligned} \tag{B.2}$$

where the complex numbers λ, α, β are defined by

$$\begin{aligned} \lambda &= (d-inc)^2 - i\ell c^2(Q-iP), \\ \alpha &= \lambda^{-1}[(d-inc)((-b+ina) + cai\ell(Q-iP)], \\ \beta &= \lambda^{-1}[(-b+ina)^2 - a^2 i\ell(Q-iP)] \end{aligned}$$

respectively. To be precise, in order to obtain again exactly the form (B.1) after the PSL$(2,\mathbb{R})$ transformation, one also has to shift $z \to z - \operatorname{Re}\alpha$ and subsequently rescale $z \to z/|\lambda|$, $t \to |\lambda|\, t$ in the supergravity solution. It is straightforward to show that the transformed parameters satisfy again the magnetic charge quantization condition.

We see that the transformation mixes the parameters in a nonlinear way. In particular, we can start from a solution with vanishing NUT-parameter and generate one with nonzero

n. These two solutions are clearly different topologically, and therefore in general the $\mathrm{PSL}(2,\mathbb{R})$ transformations are not merely diffeomorphisms from the four-dimensional point of view.

Although electromagnetic duality invariance is broken in the gauged theory due to the minimal coupling of the gravitini to the graviphoton, a generalized duality invariance was discovered in the supersymmetric subclass of the Plebański-Demiański solution, which rotates also the mass parameter into the NUT charge and vice-versa [18]. It would be interesting to see whether this duality is a consequence of the $\mathrm{PSL}(2,\mathbb{R})$ invariance of the equations (2.2)–(2.4).

C. General solution to the inhomogeneous Siklos equation

In this section we briefly review the construction of the general solution to the inhomogeneous Siklos equation (3.3) for the function $\mathcal{G}(x,y,u)$, obtained in [21]. This is a linear second order partial differential equation, and has a particular solution given by

$$\mathcal{G}_0(x,y,u) = -\frac{(\varphi'(u))^2}{16\ell^2}\left(\zeta+\bar{\zeta}\right)^4 . \tag{C.1}$$

Therefore, the generic solution reads

$$\mathcal{G}(x,y,u) = \mathcal{G}_0(x,y,u) + H(x,y,u)\,, \tag{C.2}$$

where H is the general solution to the homogeneous problem,

$$\Delta H - \frac{2}{x}\partial_x H = 0\,. \tag{C.3}$$

To characterize these solutions, one can define

$$\tilde{H} = x\int \frac{H(x,y,u)}{x}^2 \mathrm{d}x\,, \tag{C.4}$$

and the homogeneous Siklos equation reduces to

$$x^2\partial_x\left(\frac{\Delta\tilde{H}}{x}\right) = 0\,, \tag{C.5}$$

or, in other words, $\tilde{H}$ has to satisfy the Poisson equation

$$\Delta\tilde{H} = xK_{,yy}(y,u)\,, \tag{C.6}$$

where $K(y,u)$ is an arbitrary function of its arguments. The simplest example of such a function is harmonic, i.e. $\tilde{H} = f(\zeta,u) + \bar{f}(\bar{z},u)$, with $f(\zeta,u)$ an arbitrary analytic function in ζ, in which case we obtain the class of solutions

$$\mathcal{G}(\zeta,\bar{\zeta},u) = \frac{1}{2}\left(\zeta+\bar{\zeta}\right)\left(\partial f+\bar{\partial}\bar{f}\right) - \left(f+\bar{f}\right) - \frac{(\varphi'(u))^2}{16\ell^2}\left(\zeta+\bar{\zeta}\right)^4 . \tag{C.7}$$

A particular solution to (C.6) is given by $\tilde{H} = xK(y,u)$, and since it is a linear partial differential equation, it follows that its general solution is given by $\tilde{H} = xK(y,u) + f(\zeta,u) + \bar{f}(\bar{z},u)$, with $f(\zeta,u)$ a general holomorphic function in ζ.

Plugging this general solution back into the definition of H, we see that the xK term vanishes and (3.4) is the general solution to our problem, as previously stated.

D. Geometry of the lightlike case

For the geometry (3.1), the vierbein can be chosen to be

$$e^{+} = \frac{\ell^2}{x^2}\, du\,, \qquad e^{-} = dv + \frac{1}{2}\mathcal{G}\, du\,, \qquad e^{x} = \frac{\ell}{x}\, dx\,, \qquad e^{y} = \frac{\ell}{x}\, dy\,, \tag{D.1}$$

and the spin connection reads

$$\begin{aligned} \omega_{+-} &= -\frac{1}{\ell}\, e^{x}\,,\ \omega_{-x} = \frac{1}{\ell}\, e^{+}\,, & \omega_{xy} &= \frac{1}{\ell}\, e^{y}\,, \\ \omega_{-y} &= 0\,, \qquad \omega_{+x} = \frac{x^3}{2\ell^3}\, \mathcal{G}_{,x}\, e^{+} - \frac{1}{\ell}\, e^{-}\,,\ & \omega_{+x} &= \frac{x^3}{2\ell^3}\, \mathcal{G}_{,y}\, e^{+}\,. \end{aligned} \tag{D.2}$$

Finally, the supercovariant derivative becomes

$$\begin{aligned} \mathcal{D}_u &= \partial_u + \frac{\ell}{2x^2}\Gamma_{+}(1-\Gamma_x) + \frac{1}{4\ell}\mathcal{G}\Gamma_{-} + \frac{1}{4\ell}\left(x\mathcal{G}_{,x} - \mathcal{G}\right)\Gamma_{-x} + \frac{x}{4\ell}\mathcal{G}_{,y}\Gamma_{-y} + \frac{ix}{2\ell}\varphi'(u)\Gamma_{-}\Gamma_{x+}\,, \\ \mathcal{D}_v &= \partial_v + \frac{1}{2\ell}\Gamma_{-}(1-\Gamma_x)\,, \\ \mathcal{D}_x &= \partial_x - \frac{i}{\ell}\varphi(u) + \frac{1}{2x}\left(\Gamma_{+-} + \Gamma_x\right) + \frac{ix^2}{2\ell^2}\varphi'(u)\Gamma_{-}\,, \\ \mathcal{D}_y &= \partial_y + \frac{1}{2x}\Gamma_y\left(1-\Gamma_x\right) + \frac{ix^2}{2\ell^2}\varphi'(u)\Gamma_{-}\Gamma_{xy}\,. \end{aligned} \tag{D.3}$$

References

[1] K.P. Tod, *All metrics admitting supercovariantly constant spinors*, *Phys. Lett.* **B 121** (1983) 241.

[2] J.P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, *G-structures and wrapped NS5-branes*, *Commun. Math. Phys.* **247** (2004) 421 [hep-th/0205050].

[3] J.P. Gauntlett, J.B. Gutowski, C.M. Hull, S. Pakis and H.S. Reall, *All supersymmetric solutions of minimal supergravity in five dimensions*, *Class. and Quant. Grav.* **20** (2003) 4587 [hep-th/0209114];
J.P. Gauntlett and S. Pakis, *The geometry of $D = 11$ killing spinors*, *J. High Energy Phys.* **04** (2003) 039 [hep-th/0212008];
J.P. Gauntlett, J.B. Gutowski and S. Pakis, *The geometry of $D = 11$ null killing spinors*, *J. High Energy Phys.* **12** (2003) 049 [hep-th/0311112];
J.P. Gauntlett, D. Martelli and D. Waldram, *Superstrings with intrinsic torsion*, *Phys. Rev.* **D 69** (2004) 086002 [hep-th/0302158];
J.P. Gauntlett and J.B. Gutowski, *All supersymmetric solutions of minimal gauged supergravity in five dimensions*, *Phys. Rev.* **D 68** (2003) 105009 [hep-th/0304064];
J.B. Gutowski, D. Martelli and H.S. Reall, *All supersymmetric solutions of minimal supergravity in six dimensions*, *Class. and Quant. Grav.* **20** (2003) 5049 [hep-th/0306235];
J.B. Gutowski and H.S. Reall, *General supersymmetric AdS_5 black holes*, *J. High Energy Phys.* **04** (2004) 048 [hep-th/0401129];
M. Cariglia and O.A.P. Mac Conamhna, *The general form of supersymmetric solutions of $N = (1, 0)$ U(1) and SU(2) gauged supergravities in six dimensions*, *Class. and Quant. Grav.* **21** (2004) 3171 [hep-th/0402055].

[4] M.M. Caldarelli and D. Klemm, *All supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity*, *J. High Energy Phys.* **09** (2003) 019 [hep-th/0307022].

[5] M.M. Caldarelli and D. Klemm, *Supersymmetric Gödel-type universe in four dimensions*, *Class. and Quant. Grav.* **21** (2004) L17 [hep-th/0310081].

[6] S.D. Mathur, A. Saxena and Y.K. Srivastava, *Constructing 'hair' for the three charge hole*, *Nucl. Phys.* **B 680** (2004) 415 [hep-th/0311092].

[7] O. Lunin, *Adding momentum to D1-D5 system*, *J. High Energy Phys.* **04** (2004) 054 [hep-th/0404006].

[8] S. Giusto, S.D. Mathur and A. Saxena, *Dual geometries for a set of 3-charge microstates*, hep-th/0405017.

[9] C.P. Boyer and J.D. Finley, *Killing vectors in selfdual, euclidean Einstein spaces*, *J. Math. Phys.* **23** (1982) 1126.

[10] G. Guralnik, A. Iorio, R. Jackiw and S.Y. Pi, *Dimensionally reduced gravitational Chern-Simons term and its kink*, *Ann. Phys. (NY)* **308** (2003) 222 [hep-th/0305117].

[11] S. Cacciatori, D. Klemm and D. Zanon, *w_∞ algebras, conformal mechanics and black holes*, *Class. and Quant. Grav.* **17** (2000) 1731 [hep-th/9910065].

[12] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319.

[13] M.M. Caldarelli and D. Klemm, *Supersymmetry of anti-de Sitter black holes*, *Nucl. Phys.* **B 545** (1999) 434 [hep-th/9808097].

[14] A.H. Chamseddine and W.A. Sabra, *Magnetic strings in five dimensional gauged supergravity theories*, *Phys. Lett.* **B 477** (2000) 329 [hep-th/9911195].

[15] D. Klemm and W.A. Sabra, *Supersymmetry of black strings in $D = 5$ gauged supergravities*, *Phys. Rev.* **D 62** (2000) 024003 [hep-th/0001131].

[16] E. Witten, *Dynamical breaking of supersymmetry*, *Nucl. Phys.* **B 188** (1981) 513.

[17] L.J. Romans, *Supersymmetric, cold and lukewarm black holes in cosmological Einstein-Maxwell theory*, *Nucl. Phys.* **B 383** (1992) 395 [hep-th/9203018].

[18] N. Alonso-Alberca, P. Meessen and T. Ortin, *Supersymmetry of topological Kerr-Newman-Taub-NUT-AdS spacetimes*, *Class. and Quant. Grav.* **17** (2000) 2783 [hep-th/0003071].

[19] D. Grumiller and W. Kummer, *The classical solutions of the dimensionally reduced gravitational Chern-Simons theory*, *Ann. Phys. (NY)* **308** (2003) 211 [hep-th/0306036].

[20] J. Podolsky and M. Ortaggio, *Explicit kundt type-II and N solutions as gravitational waves in various type d and o universes*, *Class. and Quant. Grav.* **20** (2003) 1685 [gr-qc/0212073].

[21] S.T.C. Siklos, *Lobatchevski plane gravitational waves*, in *Galaxies, axisymmetric systems and relativity*, M.A.H. MacCallum ed., Cambridge University Press, Cambridge 1985.

[22] M. Bañados, A. Chamblin and G.W. Gibbons, *Branes, AdS gravitons and virasoro symmetry*, *Phys. Rev.* **D 61** (2000) 081901 [hep-th/9911101].

[23] P. van Nieuwenhuizen and N.P. Warner, *Integrability conditions for Killing spinors*, *Commun. Math. Phys.* **93** (1984) 277.

JHEP07(2004)061

[24] J. Podolsky, *Interpretation of the siklos solutions as exact gravitational waves in the anti-de Sitter universe*, *Class. and Quant. Grav.* **15** (1998) 719 [gr-qc/9801052].

[25] D. Brecher, A. Chamblin and H.S. Reall, *AdS/CFT in the infinite momentum frame*, *Nucl. Phys.* **B 607** (2001) 155 [hep-th/0012076].

[26] P. Breitenlohner and D.Z. Freedman, *Stability in gauged extended supergravity*, *Ann. Phys. (NY)* **144** (1982) 249.

[27] M. Cariglia, G.W. Gibbons, R. Guven and C.N. Pope, *Non-abelian pp-waves in* $D = 4$ *supergravity theories*, *Class. and Quant. Grav.* **21** (2004) 2849 [hep-th/0312256].

[28] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev.* **D 60** (1999) 064018 [hep-th/9902170].

[29] M. Cvetič et al., *Embedding AdS black holes in ten and eleven dimensions*, *Nucl. Phys.* **B 558** (1999) 96 [hep-th/9903214].

[30] S. Deser, R. Jackiw and S. Templeton, *Topologically massive gauge theories*, *Ann. Phys. (NY)* **140** (1982) 372, erratum *ibid.* **185** (1988) 406, *Ann. Phys. (NY)* **281** (2000) 409.

[31] E. Witten, $\mathrm{SL}(2,\mathbb{Z})$ *action on three-dimensional conformal field theories with abelian symmetry*, hep-th/0307041.

[32] R.G. Leigh and A.C. Petkou, $\mathrm{SL}(2,\mathbb{Z})$ *action on three-dimensional CFTS and holography*, *J. High Energy Phys.* **12** (2003) 020 [hep-th/0309177].